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On the existence of the thermomechanical terms of Akopyan and Zel'dovich in cholesteric liquid crystals

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ABSTRACT

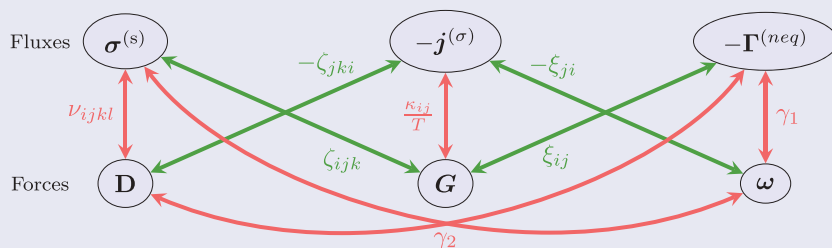
We revisit a theoretical paper of Akopyan and Zel'dovich about the thermomechanical coupling terms in nematic liquid crystals. We show that the expressions of these terms given by these authors must be corrected to satisfy the Onsager reciprocity relations, a point already stressed by Pleiner and Brand in 1987. We then extend this calculation to the cholesteric phase and show that there are no additional terms in the uniaxial approximation of this phase. Finally, we give the correspondence between the Akopyan and Zel'dovich terms and those calculated by Pleiner and Brand in 1996 by making a different choice for the forces and the fluxes in the theory.

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Thermohydrodynamic and thermomechanical couplings in a cholesteric liquid crystal

1. Introduction

In a nematic or a cholesteric phase – which is a chiral version of the previous one – several original out-of-equilibrium cross-coupling effects exist, which are forbidden in ordinary liquids. These effects are associated with the existence of a new hydrodynamic variable, the director \mathbf{n} which is the unit vector giving the mean orientation of the molecules at each point. The novelty with respect to an ordinary liquid is that the director can experience a torque, whence the name of ‘torque liquid’ sometimes given to this phase [1]. Let now suppose that a nematic (or a cholesteric) is subjected to a temperature gradient \mathbf{G} .¹ In this case, it can be shown that the molecules of the phase are submitted to a torque and a stress proportional to the temperature gradient, of general form $\Gamma_i^{(tm)} \equiv \xi_{ij} G_j$ for the torque and $-\epsilon_{ijk} \Gamma_k^{(tm)}/2 + \zeta_{ijk} G_k$ for the stress. Note that we have separated in the expression of the stress tensor the anti-symmetric part associated to the torque $-\epsilon_{ijk} \Gamma_k^{(tm)}/2$ [2] from the symmetric part $\sigma_{ij}^{(th)} \equiv \zeta_{ijk} G_k$ (with $\zeta_{ijk} = \zeta_{jik}$). In the following, we will call *thermomechanical terms* the first

coupling terms in ξ_{ij} and *thermohydrodynamic terms* the second terms in ζ_{ijk} . The former ones enter into both the torque and momentum equations whereas the second ones only enter into the momentum equation. Note that in the previous papers, the distinction between these two types of terms was not done, all of them being indifferently called thermomechanical terms.

The first cross-coupling of this type was described by Leslie in 1968 in a cholesteric phase. Leslie calculated the coupling tensors ξ_{ij} and ζ_{ijk} based on the symmetries of a cholesteric in the uniaxial approximation,² and found that ξ_{ij} (resp. ζ_{ijk}) could be expressed as a function of a unique pseudoscalar ν (resp. μ). The thermomechanical effect associated with ν was experimentally observed first indirectly by Éber and Jánossy [3,4] (see also [5,6]) and then directly by Oswald and Dequidt [7]. This effect was also numerically studied by Sarman and Laaksonen [8,9] with molecular dynamic simulations. To our knowledge, the thermohydrodynamic effect associated with μ was never observed experimentally.

Note that in a perfect (undeformed) nematic phase, the mirror symmetry imposes that $\mu = \nu = 0$ meaning that the pure Leslie thermomechanical and thermohydrodynamic effects cannot exist in this phase. However, the situation becomes different if the director field is distorted. This was shown by Akopyan and Zel'dovich in 1984 [10] who stressed that one can observe thermomechanical and thermohydrodynamic effects in a *deformed* nematic, where the gradient of the director $\nabla \mathbf{n}$ is non-zero. In this case, the coupling tensors ξ_{ij} and ζ_{ijk} must be proportional to $\nabla \mathbf{n}$ (to the smallest order). This is essential because no cross-coupling with \mathbf{G} is authorised in a undeformed nematic for obvious symmetry reasons.

By taking into account this linear dependence in $\nabla \mathbf{n}$, Akopyan and Zel'dovich calculated the structure of the coupling tensors compatible with the symmetries of a nematic, and found that ξ_{ij} can be expressed in terms of four independent scalars ξ_{1-4} , while ζ_{ijk} depends on eight independent scalars ξ_{5-12} . Several studies then focused on the consequences of these coupling terms. In particular, thermally induced flows in hybrid-oriented nematic cells were studied experimentally by Akopyan et al. [11–14] and numerically by Zakharov et al. [15]. Oswald et al. [16] showed experimentally that the coupling coefficients ξ_{1-4} play an essential role in the thermomechanical rotation of translationally invariant configurations of the cholesteric phase (where these terms should also exist). Zakharov et al. investigated the influence of the coupling terms of Akopyan and Zel'dovich on the pumping effect in a microvolume cylindrical cavity [17] and on vortical flows in bidirectionally oriented nematic cell [18]. Finally, it was shown experimentally by Jánossy et al. [19] and theoretically by Krimer and Residori [20] that these coupling terms can induce a significant lowering of the optical Freedericksz transition threshold in nematics.

On the other hand, Brand and Pleiner [21] challenged the existence of the thermomechanical and thermohydrodynamic terms of Akopyan and Zel'dovich because of a fundamental flaw in their paper [10]: indeed, a simple examination of the time-reversal transformation properties of the involved quantities shows that the coupling terms of Akopyan and Zel'dovich are *reversible*; however, it can be checked that these terms do not yield a vanishing entropy production. This means that the Onsager reciprocity relations are violated, which is indeed a serious problem. This can explain the conclusion of Pleiner and Brand in 1987 that the terms of Akopyan and Zel'dovich do not exist [21].

The irony of history is that Pleiner and Brand derived themselves very similar terms in their formulation of the nematohydrodynamics by using a different

choice for the forces and the fluxes [21,22]. The question may thus be legitimately raised as to the equivalence between the theory of Akopyan and Zel'dovich (once corrected) and that of Pleiner and Brand. In addition, it would be very interesting to extend these calculations to a cholesteric phase, knowing that several experiments were done with cholesterics.

The goal of this paper is to propose a unified framework that allows one to easily derive the phenomenological equations whatever the choice of forces and fluxes. This formalism will then be used to correct the terms given by Akopyan and Zel'dovich and show the equivalence between the two approaches proposed in the literature and generalise the calculations to a cholesteric phase.

The plan of the article is the following. In [Section 2](#), we will present the generic framework used to derive the phenomenological constitutive equations of the phase. In [Section 3](#), we will use this formalism to derive a simplified version of the phenomenological equations in the case of a cholesteric phase. In this section, only the Leslie terms will be considered. This exercise will prove to be instructive to show – as expected – that the final result does not depend on the choice of the forces and fluxes. In [Section 4](#), we will generalise the results of [Section 3](#) by including all the possible thermomechanical and thermohydrodynamic effects in a distorted cholesteric phase. The correspondence between the Akopyan and Zel'dovich terms and those of Pleiner and Brand will be given in this section. Finally, our conclusions will be drawn in [Section 5](#).

2. Generic framework

In this section, we establish the generic framework used to derive the out-of-equilibrium equations of the nematohydrodynamics. Our framework is largely inspired by the formalism of de Groot and Mazur [23] or Pleiner and Brand [22], with a notable improvement: in our approach, both reversible and irreversible fluxes can be obtained by deriving two bilinear forms of the forces, which are built in order that the fluxes automatically verify the Onsager reciprocity relations (symmetric and antisymmetric).

We start from the irreversible production of entropy, as given by Oswald and Pieranski in the incompressible approximation [2]:

$$T \overset{\circ}{\sigma} \equiv T \left(\frac{D\sigma}{Dt} + \nabla \cdot \mathbf{j}^{(\sigma)} \right) = \boldsymbol{\sigma}^{(s)} : \mathbf{D} - \Gamma^{(\text{neq})} \cdot \mathbf{w} - \mathbf{j}^{(\sigma)} \cdot \mathbf{G}, \quad (1)$$

where σ is the density of entropy, $D/Dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$ the advective derivative, $\mathbf{j}^{(\sigma)}$ the entropy flux, $\boldsymbol{\sigma}^{(s)}$ the symmetric part of the out-of-equilibrium contribution

$\sigma^{(\text{neq})}$ to the total stress tensor, \mathbf{D} the symmetric part of the velocity gradient tensor $\nabla \mathbf{v}$, $\Gamma^{(\text{neq})}$ the out-of-equilibrium contribution to the total torque exerted on the director \mathbf{n} , $w \equiv \mathbf{n} \times \dot{\mathbf{n}}$ the rotation vector of the director with $\dot{\mathbf{n}} = \mathbf{D}\mathbf{n}/Dt - \frac{1}{2}(\nabla \times \mathbf{v}) \times \mathbf{n}$ the corotational time derivative of the director, and \mathbf{G} the temperature gradient. In this equation, $\mathbf{A} : \mathbf{B} \equiv A_{ij}B_{ij}$ represents the total contraction of the two second-order tensors \mathbf{A} and \mathbf{B} . Note that this expression is the same as the one given by de Gennes and Prost [24] or Pleiner and Brand [22] in the limit $\nabla \cdot \mathbf{v} = 0$ (incompressibility condition). We also recall that the antisymmetric part of $s^{(\text{neq})}$ can be obtained from $\Gamma^{(\text{neq})}$ with the identity $\sigma_{ij}^{(a)} = -\epsilon_{ijk}\Gamma_k^{(\text{neq})}/2$ where ϵ_{ijk} is the Levi-Civita symbol [2].

The dynamical equations associated with these variables are the torque equation, which is a consequence of the angular momentum theorem [2]

$$\Gamma^{(\text{eq})} + \Gamma^{(\text{neq})} = 0,$$

the momentum conservation (or Cauchy's) equation, coming from Newton's second law [2]

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot (\boldsymbol{\sigma}^{(\text{eq})} + \boldsymbol{\sigma}^{(\text{neq})} - P\mathbf{I}),$$

and the heat transport equation [1,2]

$$\rho c_p \frac{DT}{Dt} + \nabla \cdot \mathbf{j}^{(q)} = \boldsymbol{\sigma}^{(s)} : \mathbf{D} - \Gamma^{(\text{neq})} \cdot \mathbf{w}$$

in which the term of thermal expansion has been neglected. In these equations, $\Gamma^{(\text{eq})} = \mathbf{n} \times \mathbf{h}$ is the equilibrium contribution to the total torque with $\mathbf{h} = -\delta f / \delta \mathbf{n}$ the molecular field associated with the total free energy f including elastic, electric and magnetic contributions, $\sigma_{ij}^{(\text{eq})} = -n_{k,i}(\partial f / \partial n_{k,j})$ the elastic stress tensor, ρ the density, P the pressure (which can be computed from the incompressibility condition $\nabla \cdot \mathbf{v} = 0$), c_p the specific heat capacity at fixed pressure and $\mathbf{j}^{(q)} \equiv T\mathbf{j}^{(\sigma)}$ the heat flux.

The right-hand-side of Equation (1) can be interpreted as a sum of products between *fluxes* and *forces*. Here, we assume, as de Gennes and Prost [24] did, that the choice of fluxes and forces can be made arbitrarily. We will show in the next section that the phenomenological equations do not depend on this choice. For now, let us just remark that, at equilibrium, all forces and fluxes are equal to zero, whereas, out of equilibrium, they no longer vanish – and must verify $\dot{\sigma} \geq 0$ according to the second law of thermodynamics. When the forces are not too large – i.e. when the system is not too far from equilibrium – a linear relation can be

written between the fluxes and the forces, which must verify certain symmetry properties.

More precisely, we rewrite the irreversible production of entropy as de Groot and Mazur by explicitly separating the contributions {symmetric force, antisymmetric flux} (exponent α) and {antisymmetric force, symmetric flux} (exponent β) under the transformation $t \rightarrow -t$:

$$T \dot{\sigma} = \mathbf{j}^\alpha \cdot \mathbf{f}^\alpha + \mathbf{j}^\beta \cdot \mathbf{f}^\beta.$$

In this equation, \mathbf{f}^α (\mathbf{f}^β) is a vector containing all the coordinates of the N_α (N_β) symmetric forces (antisymmetric forces) under the transformation $t \rightarrow -t$; \mathbf{j}^α (\mathbf{j}^β) is a vector containing all the coordinates of the N_α (N_β) antisymmetric fluxes (symmetric fluxes) under the transformation $t \rightarrow -t$, conjugate to the forces included in \mathbf{f}^α (\mathbf{f}^β). The values of N_α and N_β depend on the choice of forces and fluxes, but of course the total number of forces $N_\alpha + N_\beta$ must be the same as in Equation (1), i.e. 3.

The phenomenological equations can then be written as a linear relation between fluxes and forces:

$$\begin{aligned} \mathbf{j}^\alpha &= \mathbf{L}^{\alpha\alpha} \mathbf{f}^\alpha + \mathbf{L}^{\alpha\beta} \mathbf{f}^\beta \\ \mathbf{j}^\beta &= \mathbf{L}^{\beta\alpha} \mathbf{f}^\alpha + \mathbf{L}^{\beta\beta} \mathbf{f}^\beta \end{aligned} \quad (2)$$

In these equations, \mathbf{L}^{pq} ($p, q = \alpha$ or β) are matrices containing all the phenomenological coefficients. They are constrained by the symmetric and antisymmetric Onsager reciprocity relations coming from the reversibility of the microscopic equations [23]:

$$\begin{aligned} \mathbf{L}^{\alpha\alpha} &= [\mathbf{L}^{\alpha\alpha}], \\ \mathbf{L}^{\beta\beta} &= [\mathbf{L}^{\beta\beta}], \\ \mathbf{L}^{\alpha\beta} &= -[\mathbf{L}^{\beta\alpha}]. \end{aligned} \quad (3)$$

Furthermore, the Curie principle imposes that the *form* of the phenomenological Equations (2) must be invariant under the action of the symmetry group of the phase ($D_{\infty h}$ for a nematic, D_∞ for a cholesteric in the uniaxial approximation).

Note that the Onsager relations imply that only the coupling of type $\alpha\alpha$ or $\beta\beta$, called *dissipative* couplings in opposition to the so-called *reversible* couplings of type $\alpha\beta \leftrightarrow \beta\alpha$, can produce an irreversible production of entropy:

$$T \dot{\sigma} = \mathbf{L}^{\alpha\alpha} : (\mathbf{f}^\alpha \otimes \mathbf{f}^\alpha) + \mathbf{L}^{\beta\beta} : (\mathbf{f}^\beta \otimes \mathbf{f}^\beta), \quad (4)$$

where $[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i b_j$ is the dyadic product of the two vectors \mathbf{a} and \mathbf{b} . This form is expected, because the irreversible production of entropy must be invariant under the transformation $t \rightarrow -t$. We also recall that

the second law of thermodynamics $T\dot{\sigma} \geq 0$ imposes that the matrices $\mathbf{L}^{\alpha\alpha}$ and $\mathbf{L}^{\beta\beta}$ must be positive semidefinite.

In order to ease the derivation of the form of the phenomenological equations verifying the Onsager relations and the Curie principle, we use a computational trick inspired from the original paper of Akopyan and Zel'dovich [10]. This trick consists of rewriting the general system of phenomenological Equations (2) under the following form:

$$\begin{aligned} \mathbf{j}^\alpha &= \frac{\partial}{\partial \mathbf{f}^\alpha} (R_d + R_r), \\ \mathbf{j}^\beta &= \frac{\partial}{\partial \mathbf{f}^\beta} (R_d - R_r), \end{aligned} \quad (5)$$

where we defined the ‘potentials’ R_d and R_r as

$$\begin{aligned} R_d &= \frac{1}{2} (\mathbf{L}^{\alpha\alpha} : [\mathbf{f}^\alpha \otimes \mathbf{f}^\alpha] + \mathbf{L}^{\beta\beta} : [\mathbf{f}^\beta \otimes \mathbf{f}^\beta]), \\ R_r &= \mathbf{L}^{\alpha\beta} : (\mathbf{f}^\alpha \otimes \mathbf{f}^\beta) = -\mathbf{L}^{\beta\alpha} : [\mathbf{f}^\beta \otimes \mathbf{f}^\alpha]. \end{aligned}$$

A direct comparison with Equation (4) shows that $R_d = T\dot{\sigma}/2$, which can therefore be interpreted as a dissipation function associated with the couplings of types $\alpha\alpha$ and $\beta\beta$ (the so-called Rayleigh dissipation function in the literature). The new function R_r has no particular physical meaning but is a convenient quantity to easily derive the reversible couplings of types $\alpha\beta$ and $\beta\alpha$.

The constitutive relations can then be obtained by writing R_d as the sum of the most generic bilinear forms in $\{\mathbf{f}^\alpha, \mathbf{f}^\alpha\}$ and $\{\mathbf{f}^\beta, \mathbf{f}^\beta\}$ respectively, and R_r as the most generic bilinear form in $\{\mathbf{f}^\alpha, \mathbf{f}^\beta\}$. Then, using Equation (5) gives the fluxes \mathbf{j}^α and \mathbf{j}^β . Note that in their original method, Akopyan and Zel'dovich includes in the function R_d the reversible couplings of type $\alpha\beta \leftrightarrow \beta\alpha$ and do not define the function R_r ; this makes no sense – as rightfully pointed by Brand and Pleiner [21] – because these couplings are not dissipative and do not produce entropy. Our generalisation with the function R_r allows one to correct this error because the system of Equations (5) automatically verifies the Onsager relations. Note that this is also the case with the method of the Poisson brackets used by other authors such as Stark and Lubensky [25].

However, R_r and R_d must still respect the symmetries of the phase. To impose these symmetries, we use the following procedure:

- (1) First, we write down all the terms invariant under proper rotations around \mathbf{n} .
- (2) Then, we eliminate all terms which are not invariant under the transformation $\mathbf{n} \rightarrow -\mathbf{n}$. If the phase is cholesteric, we stop here.

- (3) If the phase is nematic, we eliminate all the terms which are not invariant under the reflexion in the mirror orthogonal to \mathbf{n} .

The first step is the only non-trivial one, and necessitate some results of group theory. In [Appendix A](#), we explain our method to derive the generic expression of multilinear forms invariant under proper rotations around \mathbf{n} . We will use the results of this appendix in the next sections to derive the phenomenological equations in a cholesteric phase.

To finish this section, note that our procedure can be easily generalised to include coupling terms of higher order, such as the coupling terms of Akopyan and Zel'dovich. For example, if we want to include quadratic corrective terms in the fluxes, one only needs to include in the function R_r or R_d all the possible trilinear contributions.

3. Leslie’s thermomechanical effect: equivalence of the conventions

In this section, we write down a simplified set of phenomenological equations by neglecting the hydrodynamic terms ($\mathbf{v} = \mathbf{0}$) and all the higher-order terms of Akopyan and Zel'dovich type. The complete set of phenomenological equations will be introduced in [Section 4](#).

We show the equivalence between the two different choices of fluxes and forces made by de Gennes and Prost on the one hand [24] and by Pleiner and Brand on the other hand [26]. More important, we solve the apparent contradiction between the two conventions by showing why the Leslie effect appears dissipative in one convention and reversible in the other one. Finally, we present a thought experiment that explains the difference between the phenomenological coefficients in convention.

3.1 Convention of de Gennes and Prost

As de Gennes and Prost did in their book [24], we choose as forces and fluxes³:

$$\begin{aligned} \mathbf{f}^\alpha &= \mathbf{G}, & \mathbf{j}^\alpha &= -\mathbf{j}^{(\sigma)}, \\ \mathbf{f}^\beta &= \mathbf{w}, & \mathbf{j}^\beta &= -\mathbf{I}^{(neq)}. \end{aligned}$$

First, we derive the expression of R_d as the sum of two bilinear functions of $\{\mathbf{f}^\alpha, \mathbf{f}^\alpha\}$ and $\{\mathbf{f}^\beta, \mathbf{f}^\beta\}$ respectively. According to [Appendix A.2.1](#), the terms invariant under proper rotations around \mathbf{n} of such a function are

$$(\mathbf{G}^\perp \cdot \mathbf{G}^\perp), \quad (\mathbf{G}^\parallel \mathbf{G}^\parallel), \quad (\mathbf{w} \cdot \mathbf{w}), \quad (6)$$

where we have used the following decomposition:

$$\begin{aligned} \mathbf{w} &\rightarrow \mathbf{w}^\perp \equiv \mathbf{w}, \\ \mathbf{G} &\rightarrow \mathbf{G}^\parallel \equiv \mathbf{G} \cdot \mathbf{n}, \quad \mathbf{G}^\perp \equiv \mathbf{G} - \mathbf{G}^\parallel \mathbf{n}, \end{aligned}$$

Note that we omitted $\mathbf{w}^\parallel = \mathbf{w} \cdot \mathbf{n}$ which is equal to 0 because \mathbf{w} is orthogonal to \mathbf{n} . In Equation (6), all the terms are invariant under the transformation $\mathbf{n} \rightarrow -\mathbf{n}$ and the reflection in the mirror orthogonal to \mathbf{n} , and therefore are acceptable both in a nematic and a cholesteric. We deduce that the most generic expression of R_d compatible with the symmetries of a cholesteric or a nematic can be written as

$$R_d = \frac{\gamma_1}{2} |\mathbf{w}|^2 + \frac{\kappa_\perp}{2T} |\mathbf{G}^\perp|^2 + \frac{\kappa_\parallel}{2T} (\mathbf{G}^\parallel)^2 \quad (7)$$

Second, we derive the expression of R_r as a bilinear function of $\{\mathbf{f}^\alpha, \mathbf{f}^\beta\}$. According to Appendix A.2.1, the only terms invariant under proper rotations around \mathbf{n} in this bilinear function are

$$(\mathbf{G}^\perp \cdot \mathbf{w}), \quad (\mathbf{G}^\perp \times \mathbf{w}) \cdot \mathbf{n},$$

In this equation, only $\mathbf{G}^\perp \cdot \mathbf{w}$ is invariant under the transformation $\mathbf{n} \rightarrow -\mathbf{n}$, but this contribution is not invariant under the reflection in a mirror orthogonal to \mathbf{n} ; therefore, it is allowed only in a cholesteric and not in a nematic. We deduce that the most generic expression of R_r compatible with the symmetries of a cholesteric can be written as

$$R_r = \nu (\mathbf{G}^\perp \cdot \mathbf{w}), \quad (8)$$

with $\nu = 0$ in a nematic.

From Equations (5), (7) and (8), we deduce the phenomenological equations:

$$\begin{aligned} -\mathbf{\Gamma}^{(\text{neq})} &= \gamma_1 \mathbf{w} + \nu \mathbf{G}^\perp, \\ -\mathbf{j}^{(\sigma)} &= -\nu \mathbf{w} + \frac{\kappa_\parallel}{T} \mathbf{G}^\parallel \mathbf{n} + \frac{\kappa_\perp}{T} \mathbf{G}^\perp, \end{aligned} \quad (9)$$

As expected, we recover the usual equations for the non-equilibrium torque and the heat flux, with γ_1 the rotational viscosity, κ_\parallel (κ_\perp) the thermal conductivity parallel (perpendicular) to the director, and ν the Leslie thermomechanical coefficient. Note that ν is necessarily a pseudoscalar since $\mathbf{\Gamma}^{(\text{neq})}$ is a pseudovector and \mathbf{G}^\perp is a true vector: as a consequence ν must change sign in two cholesterics enantiomer of each other. Furthermore, the Leslie thermomechanical effect appears to be non-dissipative with this choice of forces and fluxes since it is generated by the reversible potential R_r . This point was already stressed by Oswald and Pieranski in their book [2].

3.2 Convention of Pleiner and Brand

Similarly to Pleiner and Brand [22,26], we can also choose as forces and fluxes:

$$\mathbf{f}^\alpha = \begin{bmatrix} -\mathbf{\Gamma}^{(\text{neq})} \\ \mathbf{G} \end{bmatrix}, \quad \mathbf{j}^\alpha = \begin{bmatrix} \mathbf{w} \\ -\mathbf{j}^{(\sigma)} \end{bmatrix}.$$

With this convention, there are only variables of type α . For this reason, we only need to derive the expression of R_d as a bilinear function of $\{\mathbf{f}^\alpha, \mathbf{f}^\alpha\}$. As shown in Appendix A.2.1, the terms invariant under proper rotations around \mathbf{n} of such a function are

$$\begin{aligned} (\mathbf{G}^\perp \cdot \mathbf{G}^\perp), \quad (\mathbf{G}^\parallel \mathbf{G}^\parallel), \quad (\mathbf{\Gamma}^{(\text{neq})} \cdot \mathbf{\Gamma}^{(\text{neq})}), \\ (\mathbf{G}^\perp \cdot \mathbf{\Gamma}^{(\text{neq})}), \quad (\mathbf{G}^\perp \times \mathbf{\Gamma}^{(\text{neq})}) \cdot \mathbf{n}, \end{aligned} \quad (10)$$

where we have used the same type of decomposition as in the previous subsection:

$$\begin{aligned} \mathbf{\Gamma}^{(\text{neq})} &\rightarrow \mathbf{\Gamma}^\perp \equiv \mathbf{\Gamma}^{(\text{neq})}, \\ \mathbf{G} &\rightarrow \mathbf{G}^\parallel \equiv \mathbf{G} \cdot \mathbf{n}, \quad \mathbf{G}^\perp \equiv \mathbf{G} - \mathbf{G}^\parallel \mathbf{n}, \end{aligned}$$

As $\mathbf{\Gamma}^{(\text{neq})}$ is orthogonal to \mathbf{n} , we have omitted $\mathbf{\Gamma}^\parallel$ in this decomposition. In Equation (10), only the following terms are invariant under the transformation $\mathbf{n} \rightarrow -\mathbf{n}$:

$$\begin{aligned} (\mathbf{G}^\perp \cdot \mathbf{G}^\perp), \quad (\mathbf{G}^\parallel \mathbf{G}^\parallel), \quad (\mathbf{\Gamma}^{(\text{neq})} \cdot \mathbf{\Gamma}^{(\text{neq})}), \\ (\mathbf{G}^\perp \cdot \mathbf{\Gamma}^{(\text{neq})}), \end{aligned}$$

In addition, all these terms are invariant under the reflection in the mirror orthogonal to \mathbf{n} , except for $\mathbf{G}^\perp \cdot \mathbf{\Gamma}^{(\text{neq})}$ which is a pseudoscalar. We deduce that the most generic expression for R_d compatible with the symmetries of a cholesteric can be written as

$$R_d = \frac{1}{2} \left[\frac{1}{\gamma_1'} |\mathbf{\Gamma}^{(\text{neq})}|^2 + \frac{\kappa_\perp'}{T} |\mathbf{G}^\perp|^2 + \frac{\kappa_\parallel'}{T} (\mathbf{G}^\parallel)^2 + \frac{\nu'}{\gamma_1'} (\mathbf{G}^\perp \cdot \mathbf{\Gamma}^{(\text{neq})}) \right]. \quad (11)$$

In a nematic, this expression remains unchanged with $\nu' = 0$.

From Equations (5) and (11), we deduce the phenomenological equations:

$$\begin{aligned} \mathbf{w} &= -\frac{1}{\gamma_1'} \mathbf{\Gamma}^{(\text{neq})} - \frac{\nu'}{\gamma_1'} \mathbf{G}^\perp, \\ -\mathbf{j}^{(\sigma)} &= \frac{\nu'}{\gamma_1'} \mathbf{\Gamma}^{(\text{neq})} + \frac{\kappa_\parallel'}{T} \mathbf{G}^\parallel \mathbf{n} + \frac{\kappa_\perp'}{T} \mathbf{G}^\perp. \end{aligned} \quad (12)$$

The phenomenological coefficients γ_1' , $\kappa_{\parallel,\perp}'$ and ν' have the same physical meaning than in the de Gennes convention. In particular, the coefficient ν' is still a pseudoscalar. But contrary to the previous convention, the Leslie thermomechanical effect associated with ν' appears to be dissipative for this choice of forces and

fluxes. We solve this apparent contradiction in the next subsection by showing the equivalence between the two conventions and by proposing a thought experiment which gives the physical meaning to each coefficient in each convention.

3.3 Equivalence of the conventions

It is easy to check that the systems of phenomenological Equations (9) and (12) are perfectly equivalent on condition to impose the following relations between the phenomenological coefficients:

$$\begin{aligned}\gamma_1 &= \gamma'_1, \\ \nu &= \nu', \\ \kappa_{\parallel} &= \kappa'_{\parallel}, \\ \kappa_{\perp} &= \kappa'_{\perp} - \frac{T\nu'^2}{\gamma_1}.\end{aligned}$$

In particular, we notice that the thermomechanical coupling coefficient ν' introduces a correction to the thermal conductivity orthogonal to the director κ'_{\perp} . As the conductivity is a dissipative phenomenon, this correction explains why the thermomechanical coupling of Leslie is dissipative in the convention of Pleiner and Brand and reversible in the convention of de Gennes and Prost.

To better understand the difference between the two conventions, let us give the physical interpretation of the two thermal conductivities κ_{\perp} and κ'_{\perp} with the help of a thought experiment. This experiment, represented schematically in [Figure 1](#), aims to measure the thermal conductivity in a compensated cholesteric sample, in which the spontaneous twist is equal to zero. This sample is sandwiched between two plates separated by a distance d . The upper plate is thermally regulated at a temperature T_0 . The lower plate is covered with a thin conductive layer of resistance R in which an electric current I circulates. The power dissipated by Joule effect $P = RI^2$ then produces a heat flux $j^{(q)} \equiv Tj^{(\sigma)} = P/S$ between the two plates, where S is the surface of the plates. As a consequence, the lower plate heats up

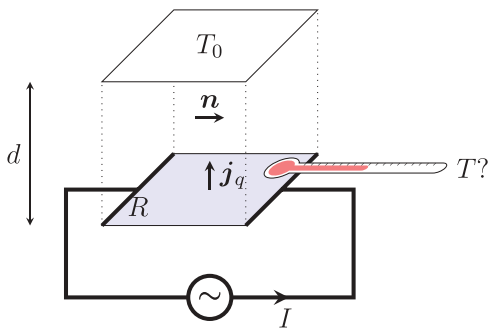


Figure 1. Thought experiment to measure the thermal conductivities κ_{\perp} et κ'_{\perp} .

and reaches a temperature T that can be measured with a thermocouple. From the knowledge of P and T , the effective thermal conductivity κ_{eff} of the liquid crystal can be obtained from the formula:

$$\kappa_{\text{eff}} = \frac{j^{(q)} d}{T - T_0} \quad (13)$$

Two situations can then be considered.

In the first situation, we assume that the director is planar and free to rotate on the surface of the plates. In this case, the system of dynamical equations given in [Section 2](#) can be easily solved with the phenomenological Equations (12) in the steady-state approximation ($j^{(q)}$ and w constant in the sample):

$$\begin{aligned}\Gamma^{(\text{eq})} &= 0, \\ w &= \frac{\nu j^{(q)}}{\gamma_1 \kappa'_{\perp}}, \\ T &= T_0 + \frac{j^{(q)} d}{\kappa'_{\perp}}.\end{aligned}$$

In this case, the texture is uniform and rotates at the angular velocity w . We deduce from Equation (13) that the effective conductivity is in this situation $\kappa_{\text{eff}} = \kappa'_{\perp}$.

In the second situation, we assume that the director is perfectly aligned with a strong magnetic field parallel to the plates. In this case, the texture cannot rotate. By solving the system of dynamical equations given in [Section 2](#) with the phenomenological Equations (9), we then find the following steady-state solution:

$$\begin{aligned}\Gamma^{(\text{eq})} &= -\frac{\nu j^{(q)}}{\kappa_{\perp}}, \\ w &= 0, \\ T &= T_0 + \frac{j^{(q)} d}{\kappa_{\perp}},\end{aligned}$$

We deduce from Equation (13) that the effective conductivity in this situation is $\kappa_{\text{eff}} = \kappa_{\perp}$. Here, $\Gamma^{(\text{eq})}$ identifies with the magnetic torque since it is the magnetic field that allows to equilibrate the Leslie thermomechanical torque.

We can therefore interpret κ_{\perp} as the thermal conductivity orthogonal to the director when the latter is **blocked**, and κ'_{\perp} as the thermal conductivity orthogonal to the director when the latter is **free to rotate**. In practice, the difference between κ_{\perp} and κ'_{\perp} is completely negligible because of the very small value of the Leslie thermomechanical coefficient in usual cholesterics [7,16,27].

In conclusion, this example shows that the choice of forces and fluxes is indifferent. Indeed, the Leslie effect is the same in the two conventions, even if it appears to be reversible or dissipative depending on the choice of forces and fluxes. This apparent contradiction is explained by the presence of a corrective term in the

dissipation when switching from one convention to the other. In the next section, we generalise these results by including the hydrodynamics and the trilinear contributions in the potentials R_d and R_r .

4. Thermohydrodynamic effect of Akopyan and Zel'dovich: generalisation in a cholesteric

In this section, we write down the complete phenomenological equations by including all viscous, thermomechanical and thermohydrodynamic couplings. We show that our approach allows to easily derive these effects – although the involved expressions are somewhat lengthy – and that the erroneous expression of the thermomechanical and thermohydrodynamic terms given in a nematic by Akopyan and Zel'dovich [10] can easily be corrected. Moreover, we show that there is no additional thermomechanical and thermohydrodynamic couplings in a cholesteric apart from the Leslie effect, in spite of the symmetry breaking $D_{\infty h} \rightarrow D_{\infty}$. Finally, we point out that the corrected terms of Akopyan and Zel'dovich are perfectly equivalent to the thermomechanical and thermohydrodynamic terms derived by Brand and Pleiner [21,22] with their own choice of forces and fluxes.

4.1 Derivation of the phenomenological equations

If we want to include hydrodynamic contributions, we must include in our choice of forces and fluxes $\sigma^{(s)}$ and \mathbf{D} , as shown in Equation (1). We thus make the following choice of forces and fluxes⁴ which is similar to the one of Akopyan and Zel'dovich:

$$\begin{aligned} \mathbf{f}^\alpha &= \mathbf{G}, & \mathbf{j}^\alpha &= -\mathbf{j}^{(\sigma)}, \\ \mathbf{f}^\beta &= \begin{pmatrix} \mathbf{w} \\ \mathbf{D} \end{pmatrix}, & \mathbf{j}^\beta &= \begin{pmatrix} -\mathbf{\Gamma}^{(\text{neq})} \\ \boldsymbol{\sigma}^{(s)} \end{pmatrix}. \end{aligned}$$

First, we derive the expression of R_d as the sum of two bilinear functions of $\{\mathbf{f}^\alpha, \mathbf{f}^\alpha\}$ and $\{\mathbf{f}^\beta, \mathbf{f}^\beta\}$ respectively. Note that we do not consider for simplicity trilinear terms of types $\{\mathbf{f}^\alpha, \mathbf{f}^\alpha, \nabla \mathbf{n}\}$ and $\{\mathbf{f}^\beta, \mathbf{f}^\beta, \nabla \mathbf{n}\}$ that would add corrective terms in $\nabla \mathbf{n}$ in the thermal conductivities and the usual viscosities. By using the results of appendices A.2.1 and A.2.2, it can be shown that the most generic expression of R_d compatible with the symmetries of a nematic or a cholesteric is

$$R_d = R_d^{(q)} + R_d^{(v)}, \quad (14)$$

with

$$\begin{aligned} R_d^{(q)} &= \frac{\kappa_{\parallel}}{2T} |G^\perp|^2 + \frac{\kappa_{\perp}}{2T} (G^\parallel)^2, \\ R_d^{(v)} &= \frac{\beta_0}{2} (d^\parallel)^2 + \frac{\beta_1}{2} |\mathbf{w}|^2 + \frac{\beta_2}{2} |\mathbf{d}^\perp|^2 + \beta_{12} (\mathbf{d}^\perp \times \mathbf{w}) \cdot \mathbf{n} + \frac{\beta_3}{2} (\mathbf{D}^\perp : \mathbf{D}^\perp), \end{aligned}$$

where we did the following decomposition⁵ for the symmetric velocity gradient tensor \mathbf{D} :

$$\begin{aligned} \mathbf{D} &\rightarrow d^\parallel \equiv \mathbf{D} : (\mathbf{n} \otimes \mathbf{n}), & d^\perp &\equiv \mathbf{D} \mathbf{n} - d^\parallel \mathbf{n}, \\ \mathbf{D}^\perp &\equiv \mathbf{D} + \frac{1}{2} (\mathbf{I} - 3\mathbf{n} \otimes \mathbf{n}) d^\parallel - \mathbf{n} \otimes d^\perp - d^\perp \otimes \mathbf{n}. \end{aligned}$$

All the terms in R_d are invariant under reflection in the mirror orthogonal to \mathbf{n} ; therefore, there are allowed both in a nematic and a cholesteric.

Second, we derive the expression of R_r as the sum of a bilinear function of $\{\mathbf{f}^\alpha, \mathbf{f}^\beta\}$ (Leslie effect) and a trilinear function of $\{\mathbf{f}^\alpha, \mathbf{f}^\beta, \nabla \mathbf{n}\}$ (Akopyan and Zel'dovich effects). Using the results of appendices A.2.1, A.2.4 and 1.2.5, it can be shown that the most generic expression of R_r compatible with the symmetries of a cholesteric is:

$$R_r = R_r^{(tm)} + R_r^{(th)} \quad (15)$$

with

$$\begin{aligned} R_r^{(tm)} &= \nu (\mathbf{w} \cdot \mathbf{G}^\perp) + \xi_1 \mathbf{n} \cdot (\mathbf{G}^\perp \times \mathbf{w}) m_1 + \xi_2 (\mathbf{G}^\perp \cdot \mathbf{w}) m_2 \\ &\quad + \xi_3 ([\mathbf{w} \times \mathbf{n}] \otimes \mathbf{G}^\perp) : \mathbf{M}^\perp + \xi_4 \mathbf{n} \cdot (\mathbf{m}^\perp \times \mathbf{w}) \mathbf{G}^\parallel, \\ R_r^{(th)} &= \mu \mathbf{n} \cdot (\mathbf{d}^\perp \times \mathbf{G}^\perp) + \xi_5 (\mathbf{d}^\perp \cdot \mathbf{G}^\perp) m_1 + \xi_6 \mathbf{n} \cdot (\mathbf{d}^\perp \times \mathbf{G}^\perp) m_2 \\ &\quad + \xi_7 (\mathbf{d}^\perp \otimes \mathbf{G}^\perp) : \mathbf{M}^\perp + \xi_8 (\mathbf{d}^\perp \cdot \mathbf{m}^\perp) \mathbf{G}^\parallel + \xi_9 (\mathbf{G}^\perp \cdot \mathbf{m}^\perp) d^\parallel \\ &\quad + \xi_{10} (d^\parallel \mathbf{G}^\parallel m_1) + \xi_{11} (\mathbf{G}^\perp \otimes \mathbf{d}^\perp) : \mathbf{D}^\perp + \xi_{12} (\mathbf{D}^\perp : \mathbf{M}^\perp) \mathbf{G}^\parallel, \end{aligned}$$

where we did the following decomposition⁶ for $\nabla \mathbf{n}$:

$$\begin{aligned} \nabla \mathbf{n} &\rightarrow m_1 \equiv \nabla \cdot \mathbf{n}, & m_2 &\equiv \mathbf{n} \cdot \nabla \times \mathbf{n}, & m^\perp &\equiv \frac{1}{2} (\mathbf{n} \cdot \nabla) \mathbf{n}, \\ & & \mathbf{M}^\perp &\equiv \frac{1}{2} ([\nabla \mathbf{n}] + [\nabla \mathbf{n}] - m_1 \mathbf{I}^\perp - 2 \mathbf{n} \otimes \mathbf{m}^\perp - 2 \mathbf{m}^\perp \otimes \mathbf{n}). \end{aligned}$$

In the expressions of $R_r^{(tm)}$ and $R_r^{(th)}$, the only terms which are not invariant under a reflection in the mirror orthogonal to \mathbf{n} is $\mathbf{w} \cdot \mathbf{G}^\perp$ and $\mathbf{n} \cdot (\mathbf{d}^\perp \times \mathbf{G}^\perp)$, which implies that ν and μ are pseudoscalars, with $\nu = \mu = 0$ in a nematic. All the other terms – invariant under reflection – are therefore allowed both in a nematic and a cholesteric.

Using Equations (5), (14) and (15), we finally deduce the complete phenomenological equations in a cholesteric:

$$\begin{aligned} -j_i^{(\sigma)} &= \frac{\kappa_{ij} G_j}{T} - \xi_{ji} w_j - \zeta_{jki} D_{jk}, \\ -\Gamma_i^{(\text{neq})} &= \xi_{ij} G_j + \beta_1 w_i + \beta_{ijk} D_{jk}, \\ \sigma_{ij}^{(s)} &= \zeta_{ijk} G_k + \beta_{kij} w_k + \nu_{ijkl} D_{kl}, \end{aligned} \quad (16)$$

where the complete expressions of the coupling tensors κ_{ij} , ξ_{ij} , ζ_{ijki} , β_{ijk} and ν_{ijkl} are given in Appendix B. Note that the Onsager relations can be directly checked in the last three equations. In particular, we see that ξ_{ij} and ζ_{ijk} are associated, as expected, with reversible effects since these terms are eliminated in the entropy production $\dot{\sigma}$ (this can be shown by injecting the expressions given above into Equation (1)).

We can now give the physical interpretation of each tensor.

The coupling tensor κ_{ij} is the usual tensor of thermal conductivity. The second law of thermodynamics $\dot{\sigma} \geq 0$ imposes that $\kappa_{\perp, \parallel} \geq 0$.

The coupling tensors β_1 , β_{ijk} (containing β_{12}) and ν_{ijkl} (containing $\beta_{0,2,3}$) correspond to the usual viscous dissipation. Akopyan and Zel'dovich give the same expressions as us for these effects. It can be easily checked that the coefficients β_i are connected with the Leslie viscosities α_{1-6} by the following relations:

$$\begin{aligned}\beta_0 &= \alpha_1 + \frac{3\alpha_4}{2} + \alpha_5 + \alpha_6 \\ \beta_1 &= \alpha_3 - \alpha_2 \equiv \gamma_1 \\ \beta_{12} &= \alpha_3 + \alpha_2 \equiv \gamma_2 \\ \beta_2 &= 2\alpha_4 + \alpha_5 + \alpha_6 \\ \beta_3 &= \alpha_4\end{aligned}$$

Furthermore, it should be noted that in our approach, we only have five independent coefficients instead of the six α_i . This is expected because here the Onsager relations are automatically verified, whereas in the Leslie approach these relations impose the so-called Parodi identity $\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5$ [29] (5 coefficients \leftrightarrow 6 coefficients + 1 linear relation). This point was already underlined by several authors [10,22,24,29]. Finally, the second principle impose that $\beta_{0,1,2,3} \geq 0$ and $[\beta_{12}]^2 \leq \beta_1 + \beta_2$.

As for ξ_{ij} , it is the coupling tensor associated with the thermomechanical effects of Leslie (coefficient ν) and Akopyan and Zel'dovich (coefficients ξ_{1-4}). Akopyan and Zel'dovich gives the same expressions as us for these effects in a nematic but with the wrong sign for the heat flux \mathbf{q}^{TM} in Equation A.6 of Reference [10]. Here, we get the good sign because our approach automatically verifies the anti-symmetric Onsager reciprocity relations associated with reversible effects.

ζ_{ijk} is the coupling tensor associated with the thermohydrodynamic effects of Leslie (coefficient μ) and Akopyan and Zel'dovich (coefficients ξ_{5-12}). Again, Akopyan and Zel'dovich give the same expression as us for these effects in a nematic, apart from the wrong sign in the heat flux which is corrected here.

We underline once again that in a cholesteric, the only couplings which are not invariant under a reflection in the mirror orthogonal to \mathbf{n} are those associated with the Leslie thermomechanical and thermohydrodynamic coefficients ν and μ . This means that all the other effects are allowed both in a nematic and a cholesteric. In particular, the corrected expression of the coupling terms of Akopyan and Zel'dovich – originally derived in a nematic – is the same in a cholesteric.

4.2 Equivalence with the formalism of Pleiner and Brand

We conclude this section by giving the equivalence between our phenomenological equations and those derived by Pleiner and Brand [21,22,26] with their own choice of forces and flux. With the following choice of forces and fluxes:

$$\begin{aligned}\mathbf{f}^\alpha &= \begin{bmatrix} -\Gamma^{(neq)} \\ \mathbf{G} \end{bmatrix}, & \mathbf{j}^\alpha &= \begin{bmatrix} w \\ -\mathbf{j}^{(\sigma)} \end{bmatrix}, \\ \mathbf{f}^\beta &= \mathbf{D} & \mathbf{j}^\beta &= \sigma^{(s)},\end{aligned}$$

these authors give the following phenomenological equations, obtained with a method different from ours:

$$\begin{aligned}-j_i^{(\sigma)} &= \frac{\kappa'_{ij} G_j}{T} - \Pi_{ji} \left(-\Gamma_j^{(neq)} \right) + \Psi_{jki} D_{jk}, \\ w_i &= -\Pi_{ij} G_j + \frac{(-\Gamma_i^{(neq)})}{\gamma_1} - \lambda_{ijk} D_{jk}, \\ \sigma_{ij}^{(s)} &= -\Psi_{ijk} G_k + \lambda_{kij} \left(-\Gamma_k^{(neq)} \right) + \nu'_{ijkl} D_{kl}.\end{aligned}\tag{17}$$

The complete expressions of the coupling tensors κ'_{ij} , Π_{ij} , Ψ_{ijk} , λ_{ijk} and ν'_{ijkl} are given in Appendix B. The previous equation is equivalent to our phenomenological equations given in Equation (16) on condition to impose the following relations:

$$\begin{aligned}\kappa_{ij} &= \kappa'_{ij} - \gamma'_1 T \Pi_{ki} \Pi_{kj}, \\ \xi_{ij} &= \gamma'_1 \Pi_{ij}, \\ \zeta_{ijk} &= -\Psi_{ijk} + \gamma'_1 \Pi_{lk} \lambda_{lij}, \\ \beta_1 &= \gamma'_1, \\ \nu_{ijkl} &= \nu'_{ijkl} + \gamma'_1 \lambda_{mij} \lambda_{mkl}.\end{aligned}\tag{18}$$

Note that these equalities are verified if and only if the left and right hand sides have exactly the same tensorial form. This is true for the last four equalities – as shown in Appendix B.3 in which the full correspondence between the phenomenological coefficients is given – but wrong in the first equality, since $\Pi_{ki} \Pi_{kj}$ depends on $\nabla \mathbf{n}$ whereas the thermal conductivity tensors κ_{ij} and κ'_{ij} were computed without accounting for corrective terms in $\nabla \mathbf{n}$. Technically speaking, the only way to get an exact correspondence between the two conventions would be to include these corrective terms. However, we emphasise that such terms are extremely small and negligible as long as the director field is distorted over a length scale much larger than the intermolecular distance (a basic assumption in hydrodynamics). The fact that $\gamma'_1 T \Pi_{ki} \Pi_{kj} \ll \kappa_{ij}$ can also be checked directly by taking the typical values of thermomechanical coefficients given in the literature [16]. For this reason, one can write that $\kappa'_\perp = \kappa_\perp$ and $\kappa'_\parallel \approx \kappa_\parallel$ to within an excellent approximation, which finishes to show the equivalence between the two conventions.

5. Conclusion

We hope to have clarified the situation on the existence of the thermomechanical and thermohydrodynamic terms of Akopyan and Zel'dovich in a nematic liquid crystal. To achieve this goal, we introduced a new function equivalent to the classical dissipation function to calculate the reversible couplings. We then used this general formalism to derive the constitutive equations of a nematic or a cholesteric liquid crystal. Doing this, we realised that the choice of forces and fluxes can influence the apparent dissipative/reversible character of a cross-coupling in spite of the strict equivalence of the calculated phenomenological equations. This was explicitly shown on the example of the Leslie coupling only present in cholesterics. We then used this general formalism to correct the sign errors in the original paper of Akopyan and Zel'dovich (the same error is also made by Sonnet and Virga in their book published in 2012 [30] when they derive the Leslie effect in cholesterics). Note that, fortunately, these sign errors only appear in the corrections to the heat flux, extremely small and always negligible – and thus neglected – in practice. More important, we showed that the Akopyan and Zel'dovich terms were strictly equivalent to those derived by Brand and Pleiner by making another choice for the forces and the fluxes. The equivalence relations between the terms of Akopyan and Zel'dovich and the ones of Brand and Pleiner are also given in the paper. Last but not least, we generalised the calculations to the cholesteric phase and showed that there are no additional terms in this phase (out of the Leslie terms, only present in cholesterics).

Notes

1. A gradient of an electric or chemical potential would produce the same effects, as was stressed by de Gennes in his book on liquid crystals [24].
2. In this case, the cholesteric phase is still described by a single director \mathbf{n} which rotates along the helical axis.
3. Note that in Reference [24], the role of the fluxes and forces is reversed because de Gennes prefers to write the forces as a function of the fluxes, in opposition to our choice of writing the fluxes as a function of the forces. Nevertheless, our choice of forces and fluxes is completely equivalent to the convention of de Gennes because the swapping transformation $\mathbf{j}^\alpha \leftrightarrow \mathbf{j}^\beta$ and $\mathbf{j}^\beta \leftrightarrow \mathbf{j}^\alpha$ preserves the Onsager relations and the time-reversal behaviour of each quantity.
4. Note that in this equation, the second-order tensors \mathbf{D} and $\boldsymbol{\sigma}^{(s)}$ are by convention flattened in one-dimensional vectors.
5. Several terms of the decomposition presented in the appendix were eliminated here because \mathbf{D} is symmetric and traceless ($\nabla \cdot \mathbf{v} = 0$).

6. Several terms of the decomposition presented in the appendix were eliminated here because $\mathbf{n} \cdot \mathbf{n} = 1$.

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Appendices

Appendix A. Generic expression of a bi(tri)linear form invariant under proper rotations around \mathbf{n}

In this appendix, we show how to compute the most generic bi(tri)linear form invariant under proper rotations around the director \mathbf{n} . First, we recall some important results of group theory. Then, we use these results to compute five types of invariant functions: a bilinear form of two vectors, a bilinear form of one vector and one second-order tensor, a bilinear form of two second-order tensors, a trilinear form of two vectors and a second-order tensor, and finally a trilinear form of one vector and two second-order tensors.

A.1 A few reminders on group theory

A.1.1 Action of a proper rotation on a vector

First, we recall that the *action* of a proper rotation of θ around \mathbf{n} on a vector \mathbf{u} is *represented* by the matrix $\mathbf{R}(\theta)$:

$$\mathbf{u}' = \mathbf{R}(\theta)\mathbf{u}, \quad (\text{A1})$$

where \mathbf{u}' is the transformed vector. In an orthonormal basis \mathcal{B} whose third axis coincides with \mathbf{n} , the coordinates of $\mathbf{R}(\theta)$ are given by

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We also recall that finding the irreducible representations for the group of proper rotations is equivalent to decompose $\mathbf{R}(\theta)$ in a direct sum of non-diagonalisable square matrices, i.e. $\exists \mathbf{P}$ invertible such as $\forall \theta$,

$$\begin{aligned} \mathbf{P}\mathbf{R}(\theta)\mathbf{P}^{-1} &= \Delta^\alpha(\theta) \oplus \Delta^\beta(\theta) \oplus \dots \\ &\equiv \begin{pmatrix} \Delta^\alpha(\theta) & & & \\ & \Delta^\beta(\theta) & & \\ & & \ddots & \end{pmatrix}. \end{aligned}$$

As $\mathbf{R}(\theta)$ is diagonalisable in \mathbb{C} , this decomposition corresponds here to three 1×1 complex matrices:

$$\mathbf{P}\mathbf{R}(\theta)\mathbf{P}^{-1} = \Delta^{-1}(\theta) \oplus \Delta^1(\theta) \oplus \Delta^0(\theta),$$

with the matrices $\Delta^\alpha(\theta)$ and \mathbf{P} given by

$$\begin{aligned} \Delta^\alpha(\theta) &= (e^{i\alpha\theta}) \\ \mathbf{P} &= \begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

The main advantage of introducing the irreducible representations $\Delta^\alpha(\theta)$ is that the transformation property (A1) can be rewritten under a very simple form in the diagonalisation basis of $\mathbf{R}(\theta)$. Indeed, if we project a vector \mathbf{u} in this basis:

$$\begin{pmatrix} \tilde{u}^{-1} \\ \tilde{u}^1 \\ \tilde{u}^0 \end{pmatrix} \equiv \mathbf{P}\mathbf{u} = \begin{pmatrix} u_1 - iu_2 \\ u_1 + iu_2 \\ u_3 \end{pmatrix}, \quad (\text{A2})$$

Equation (A1) can then be rewritten as

$$[\tilde{u}^\alpha] = \Delta^\alpha(\theta)\tilde{u}^\alpha,$$

where $\alpha = -1, 0, 1$ and $\Delta^\alpha(\theta) \equiv \Delta_{11}^\alpha(\theta)$. We also emphasise that the coordinates of \mathbf{u} in the diagonalisation basis verify the following complex conjugation property:

$$\tilde{u}^{-\alpha} = [\tilde{u}^\alpha]^* \quad (\text{A3})$$

for $\alpha = -1, 0, 1$.

A.1.2 Action of a proper rotation on a second-order tensor

Second, we recall that the action of a proper rotation of θ around \mathbf{n} on a second-order tensor \mathbf{M} is represented by the following operation:

$$\mathbf{M}' = \mathbf{R}(\theta)\mathbf{M}\mathbf{R}(\theta),$$

or equivalently by the following matrix-vector product:

$$\begin{pmatrix} M'_{11} \\ M'_{12} \\ M'_{13} \\ M'_{21} \\ \vdots \\ M'_{33} \end{pmatrix} = \mathbf{T}(\theta) \begin{pmatrix} M_{11} \\ M_{12} \\ M_{13} \\ M_{21} \\ \vdots \\ M_{33} \end{pmatrix}, \quad (\text{A4})$$

where $\mathbf{T}(\theta)$ is defined as the Kronecker product of $\mathbf{R}(\theta)$ with itself:

$$\begin{aligned} [\mathbf{T}(\theta)] &\equiv [\mathbf{R}(\theta) \otimes \mathbf{R}(\theta)] \\ &\equiv \begin{pmatrix} R_{11}(\theta)\mathbf{R}(\theta) & R_{12}(\theta)\mathbf{R}(\theta) & R_{13}(\theta)\mathbf{R}(\theta) \\ R_{21}(\theta)\mathbf{R}(\theta) & R_{22}(\theta)\mathbf{R}(\theta) & R_{33}(\theta)\mathbf{R}(\theta) \\ R_{31}(\theta)\mathbf{R}(\theta) & R_{32}(\theta)\mathbf{R}(\theta) & R_{33}(\theta)\mathbf{R}(\theta) \end{pmatrix}. \end{aligned}$$

$\mathbf{T}(\theta)$ has the following decomposition into irreducible representations:

$$\begin{aligned} \mathbf{Q}\mathbf{T}(\theta)\mathbf{Q}^{-1} &= \Delta^{-2}(\theta) \oplus \Delta^2(\theta) \oplus \Delta^{-1}(\theta) \oplus \Delta^{-1}(\theta) \oplus \Delta^1(\theta) \\ &\quad \oplus \Delta^1(\theta) \oplus \Delta^0(\theta) \oplus \Delta^0(\theta) \oplus \Delta^0(\theta), \end{aligned}$$

with the matrices \mathbf{Q} and $\Delta^\alpha(\theta)$ given by

$$\begin{aligned} \mathbf{Q} &= \begin{pmatrix} 1 & -i & 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 1 & i & 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 1 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & i & 0 \\ 0 & 0 & 1 & 0 & 0 & i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \Delta^\alpha(\theta) &= (e^{i\alpha\theta}) \end{aligned}$$

Note that we find the same irreducible representations as before – but with different multiplicities – plus two irreducible representations $\Delta^{\pm 2}(\theta)$.

Similarly to the case of vectors, the transformation property (A4) has a very simple form in the diagonalisation basis of $\mathbf{T}(\theta)$. Indeed, if we project a tensor \mathbf{M} in this basis:

$$\begin{pmatrix} \tilde{M}_1^{-2} \\ \tilde{M}_2^{-2} \\ \tilde{M}_1^{-1} \\ \tilde{M}_2^{-1} \\ \tilde{M}_1^0 \\ \tilde{M}_2^0 \\ \tilde{M}_3^0 \end{pmatrix} \equiv \mathbf{Q} \begin{pmatrix} M_{11} \\ M_{12} \\ M_{13} \\ M_{21} \\ M_{22} \\ M_{23} \\ M_{31} \\ M_{32} \\ M_{33} \end{pmatrix} = \begin{pmatrix} (M_{11} - M_{22}) - i(M_{12} + M_{21}) \\ (M_{11} - M_{22}) + i(M_{12} + M_{21}) \\ M_{31} - iM_{32} \\ M_{13} - iM_{23} \\ M_{31} + iM_{32} \\ M_{13} + iM_{23} \\ M_{11} + M_{22} + M_{33} \\ M_{12} - M_{21} \\ M_{33} \end{pmatrix}, \quad (\text{A5})$$

Equation (A4) can then be rewritten as

$$[\tilde{M}_r^\alpha] = \Delta^\alpha(\theta) \tilde{M}_r^\alpha,$$

where $\alpha = -2 \dots 2$, $r = 1 \dots (3 - |\alpha|)[0, 1]$ and as before $\Delta^\alpha(\theta) \equiv \Delta_{11}^\alpha(\theta)$. Here, r represents the multiplicity index inside a given irreducible representation. We also underline that this decomposition verifies the following complex conjugation property:

$$\tilde{M}_r^{-\alpha} = [\tilde{M}_r^\alpha]^*. \quad (\text{A6})$$

A.1.3 Orthogonality relations

Last, we recall that the irreducible representations $\Delta^\alpha(\theta)$ verify powerful orthogonality relations coming from the Peter-Weyl theorem:

$$\frac{1}{2\pi} \int_0^{2\pi} \Delta_{jk}^\alpha(\theta) \Delta_{lm}^\beta(\theta)^* d\theta = \delta_{\alpha\beta} \delta_{jl} \delta_{km}. \quad (\text{A7})$$

These relations can be greatly simplified in our case because the dimension of the matrices $\Delta^\alpha(\theta)$ is 1×1 . Note that the $\Delta^\alpha(\theta)$ verifies two other interesting properties:

$$\Delta^\alpha(\theta)^* = \Delta^{-\alpha}(\theta),$$

$$\Delta^\alpha(\theta) \Delta^\beta(\theta) = \Delta^{\alpha+\beta}(\theta),$$

which can be used to generalise Equation (A7):

$$\frac{1}{2\pi} \int_0^{2\pi} \Delta^{\alpha_1}(\theta) \dots \Delta^{\alpha_N}(\theta) d\theta = \delta_{0(\alpha_1 + \dots + \alpha_N)}. \quad (\text{A8})$$

A.2 Generic expressions of multilinear invariant forms

We now use the previous theoretical results to compute the generic expressions of five types of multilinear form invariant under proper rotations around \mathbf{n} .

A.2.1 Bilinear form of two vectors

The most generic bilinear form of two vectors \mathbf{u} and \mathbf{v} can be written as

$$g = \sum_{j,k=1}^3 L_{jk} u_j v_k,$$

or equivalently in the diagonalisation basis of $\mathbf{R}(\theta)$:

$$g = \sum_{\alpha,\beta=-1}^1 \Lambda^{\alpha\beta} \tilde{u}^\alpha \tilde{v}^\beta.$$

From $g \in \mathbb{R}$ and Equation (A3), it can be shown that the coefficients $\Lambda^{\alpha\beta}$ must verify the following relation:

$$[\Lambda^{\alpha\beta}]^* = \Lambda^{-\alpha-\beta}. \quad (\text{A9})$$

Furthermore, if we impose that g is invariant under all proper rotations around \mathbf{n} , then the following relation must be verified for all θ :

$$g = \sum_{\alpha,\beta=-1}^1 \Lambda^{\alpha\beta} [\Delta^\alpha(\theta) \tilde{u}^\alpha] [\Delta^\beta(\theta) \tilde{v}^\beta].$$

By averaging over all θ , we get

$$g = \sum_{\alpha,\beta=-1}^1 \frac{1}{2\pi} \int_0^{2\pi} \Lambda^{\alpha\beta} [\Delta^\alpha(\theta) \tilde{u}^\alpha] [\Delta^\beta(\theta) \tilde{v}^\beta] d\theta.$$

Using the orthogonality relations in Equation (A8), we compute from the last equation:

$$g = \sum_{\alpha,\beta=-1}^1 \Lambda^{\alpha\beta} \tilde{u}^\alpha \tilde{v}^\beta \delta_{0(\alpha+\beta)}$$

$$= \Lambda^{-11} \tilde{u}^{-1} \tilde{v}^1 + \Lambda^{1-1} \tilde{u}^1 \tilde{v}^{-1} + \Lambda^{00} \tilde{u}^0 \tilde{v}^0.$$

According to Equation (A9), we can define

$$\Lambda^{1-1} = [\Lambda^{-11}]^* \equiv (\lambda_1 + i\lambda_2)/2,$$

$$\Lambda^{00} \equiv \eta,$$

where the coefficients λ_i and η are real. Using these definitions and Equation (A3), we can rewrite the expression of g as

$$g = \lambda_1 \text{Re}\{\tilde{u}^{-1} \tilde{v}^1\} + \lambda_2 \text{Im}\{\tilde{u}^{-1} \tilde{v}^1\} + \eta \tilde{u}^0 \tilde{v}^0,$$

or equivalently using Equation (A2):

$$g = \lambda_1 (\mathbf{u}^\perp \cdot \mathbf{v}^\perp) + \lambda_2 (\mathbf{u}^\perp \times \mathbf{v}^\perp) \cdot \mathbf{n} + \eta (\mathbf{u}^\parallel \mathbf{v}^\parallel),$$

where we have defined the scalars:

$$u^\parallel \equiv u_3 = \mathbf{u} \cdot \mathbf{n},$$

$$v^\parallel \equiv v_3 = \mathbf{v} \cdot \mathbf{n},$$

and the vectors:

$$\mathbf{u}^\perp \equiv \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \mathbf{u} - u^\parallel \mathbf{n},$$

$$\mathbf{v}^\perp \equiv \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \mathbf{v} - v^\parallel \mathbf{n},$$

We conclude by asserting that the most generic bilinear form of two vectors \mathbf{u} and \mathbf{v} invariant under proper rotations

around \mathbf{n} can be written as a linear combination of the following terms:

$$(\mathbf{u}^\perp \cdot \mathbf{v}^\perp), \quad (\mathbf{u}^\perp \times \mathbf{v}^\perp) \cdot \mathbf{n}, \quad (\mathbf{u}^\parallel \mathbf{v}^\parallel). \quad (\text{A10})$$

In the next subsections, we will give similar results for different types of bilinear and trilinear forms. We will omit the associated proofs, as they are very similar to the one given here.

A.2.2 Bilinear form of one vector and one second-order tensor

The most generic bilinear form of one vector \mathbf{u} and one second-order tensor \mathbf{M} invariant under proper rotations around \mathbf{n} can be written as a linear combination of the following terms:

$$(\mathbf{u}^\perp \cdot \mathbf{m}^{\perp r}), \quad (\mathbf{u}^\perp \times \mathbf{m}^{\perp r}) \cdot \mathbf{n}, \quad (\mathbf{u}^\parallel m_r), \quad (\text{A11})$$

where we have defined the scalars m_r ($r = 1, 2, 3$) as

$$\begin{aligned} m_1 &\equiv M_{11} + M_{22} + M_{33} &= \text{Tr } \mathbf{M}, \\ m_2 &\equiv M_{12} - M_{21} &= \epsilon_{ijk} n_i M_{jk}, \\ m_3 &\equiv M_{33} &= \mathbf{M} : (\mathbf{n} \otimes \mathbf{n}), \end{aligned}$$

and the vectors $\mathbf{m}^{\perp r}$ ($r = 1, 2$) as

$$\mathbf{m}^{\perp 1} \equiv \begin{pmatrix} M_{31} \\ M_{32} \\ 0 \end{pmatrix} = \mathbf{M}^\top \mathbf{n} - M^\parallel \mathbf{n}.$$

$$\mathbf{m}^{\perp 2} \equiv \begin{pmatrix} M_{13} \\ M_{23} \\ 0 \end{pmatrix} = \mathbf{M} \mathbf{n} - M^\parallel \mathbf{n}.$$

\mathbf{u}^\perp and u^\parallel were already defined in the previous subsection.

A.2.3 Bilinear form of two second-order tensors

The most generic bilinear form g of two second-order tensors \mathbf{M} and \mathbf{D} invariant under proper rotations around \mathbf{n} can be written as a linear combination of the following terms:

$$\begin{aligned} (\mathbf{M}^\perp : \mathbf{D}^\perp), \quad (\mathbf{M}^\perp : [\mathbf{XD}^\perp]), \\ (\mathbf{m}^{\perp r} \cdot \mathbf{d}^{\perp s}), \quad (\mathbf{m}^{\perp r} \times \mathbf{d}^{\perp s}) \cdot \mathbf{n}, \quad (m_r d_s) \end{aligned} \quad (\text{A12})$$

where we have defined the tensor \mathbf{M}^\perp as

$$\mathbf{M}^\perp \equiv \frac{1}{2} \begin{pmatrix} M_{11} - M_{22} & M_{12} + M_{21} & 0 \\ M_{12} + M_{21} & M_{22} - M_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} &= \frac{1}{2} (M + M - [m_1 - m_3] \mathbf{I}^\perp - 2m_3 [\mathbf{n} \otimes \mathbf{n}] \\ &\quad - [m^{\perp 1} + m^{\perp 2}] \otimes \mathbf{n} - \mathbf{n} \otimes [m^{\perp 1} + m^{\perp 2}]). \end{aligned}$$

and the ‘ \mathbf{n} -cross’ operator \mathbf{X} as

$$X_{ij} = \epsilon_{ijk} n_k.$$

\mathbf{D}^\perp , $d^{\perp r}$ and d_r are defined similarly to \mathbf{M}^\perp , $m^{\perp r}$ and m_r .

A.2.4 Trilinear form of two vectors and one second-order tensor

The most generic trilinear form of two vectors $\{u, v\}$ and a second-order tensor \mathbf{M} invariant under proper rotations around \mathbf{n} can be written as a linear combination of the following terms:

$$\begin{aligned} (\mathbf{u}^\perp \cdot \mathbf{v}^\perp) m_r, \quad ([\mathbf{u}^\perp \times \mathbf{v}^\perp] \cdot \mathbf{n}) m_r, \quad (u^\parallel v^\parallel) m_r, \\ (\mathbf{u}^\perp \cdot \mathbf{m}^{\perp r}) v^\parallel, \quad ([\mathbf{u}^\perp \times \mathbf{m}^{\perp r}] \cdot \mathbf{n}) v^\parallel, \\ (\mathbf{v}^\perp \cdot \mathbf{m}^{\perp r}) u^\parallel, \quad ([\mathbf{v}^\perp \times \mathbf{m}^{\perp r}] \cdot \mathbf{n}) u^\parallel, \\ (\mathbf{u}^\perp \otimes \mathbf{v}^\perp) : \mathbf{M}^\perp, \quad ([\mathbf{u}^\perp \times \mathbf{v}^\perp] \otimes \mathbf{v}^\perp) : \mathbf{M}^\perp, \end{aligned} \quad (\text{A13})$$

with the same conventions as in the previous subsections.

A.2.5 Trilinear form of one vector and two second-order tensors

The most generic trilinear form of one vector \mathbf{u} and two second-order tensors $\{M, D\}$ invariant under proper rotations around \mathbf{n} can be written as a linear combination of the following terms:

$$\begin{aligned} (\mathbf{u}^\perp \cdot \mathbf{m}^{\perp s}) d_r, \quad ([\mathbf{u}^\perp \times \mathbf{m}^{\perp s}] \cdot \mathbf{n}) d_r, \quad (u^\parallel m_r d_s), \\ (\mathbf{u}^\perp \cdot \mathbf{d}^{\perp s}) m_r, \quad ([\mathbf{u}^\perp \times \mathbf{d}^{\perp s}] \cdot \mathbf{n}) m_r, \\ (\mathbf{u}^{\perp r} \cdot \mathbf{d}^{\perp s}) u^\parallel, \quad ([\mathbf{m}^{\perp r} \times \mathbf{d}^{\perp s}] \cdot \mathbf{n}) u^\parallel, \\ (\mathbf{m}^{\perp r} \otimes \mathbf{u}^\perp) : \mathbf{D}^\perp, \quad ([\mathbf{m}^{\perp r} \times \mathbf{n}] \otimes \mathbf{u}^\perp) : \mathbf{D}^\perp, \\ (\mathbf{d}^{\perp r} \otimes \mathbf{u}^\perp) : \mathbf{M}^\perp, \quad ([\mathbf{d}^{\perp r} \times \mathbf{n}] \otimes \mathbf{u}^\perp) : \mathbf{M}^\perp, \\ (\mathbf{M}^\perp : \mathbf{D}^\perp) u^\parallel, \quad (\mathbf{M}^\perp : [\mathbf{XD}^\perp]) u^\parallel, \end{aligned} \quad (\text{A14})$$

with the same conventions as in the previous subsections.

Appendix B. Expression of the coupling tensors

In this appendix, we give the full expressions of the coupling tensors used in the phenomenological equations and the equivalence between the coupling tensors in each convention. In the following, the tensor δ_{ij}^\perp denotes the transverse Kronecker delta:

$$\delta_{ij}^\perp = \delta_{ij} - n_i n_j$$

B.1 Convention of Akopyan and Zel’dovich

The tensor of thermal conductivity κ_{ij} is given by

$$\kappa_{ij} = \kappa_\perp \delta_{ij}^\perp + \kappa_\parallel n_i n_j.$$

The coupling tensor ξ_{ij} associated with the thermomechanical effects of Leslie, Akopyan and Zel’dovich is given by

$$\begin{aligned} \xi_{ij} &= [\nu + \xi_2 (\epsilon_{kpq} n_k n_{q,p})] \delta_{ij}^\perp + (\xi_1 - \frac{\xi_3}{2}) n_{l,i} n_k \epsilon_{ikj} \\ &\quad + (\frac{\xi_4 - \xi_3}{2}) (\epsilon_{ikp} n_k n_{q,p,q}) n_j + \frac{\xi_3}{2} \epsilon_{ikp} n_k (n_{p,j} + n_{j,p}). \end{aligned}$$

The coupling tensor ζ_{ijk} associated with the thermohydrodynamic effects of Leslie, Akopyan and Zel’dovich is given by

$$\begin{aligned}
\zeta_{ijk} &= \left[\frac{\mu}{2} - \frac{\xi_6}{2} (\epsilon_{ikp} n_k n_q n_{p,q}) \right] (n_i \epsilon_{jlk} + n_j \epsilon_{ilk}) n_l \\
&+ \left(\frac{\xi_5}{2} - \frac{\xi_7}{4} \right) (n_i \delta_{jk} + n_j \delta_{ik}) n_{l,l} \\
&+ \frac{\xi_7}{4} [n_i (n_{k,j} + n_{j,k}) + n_j (n_{k,i} + n_{i,k})] \\
&- \left(\frac{\xi_5}{4} - \frac{\xi_8}{4} + \frac{\xi_{11}}{4} + \frac{\xi_{12}}{2} \right) (n_i n_{j,l} + n_j n_{i,l}) n_l n_k \\
&- \left(\xi_5 - \frac{\xi_7}{2} - \xi_{10} - \frac{\xi_{12}}{2} \right) n_{l,l} n_i n_j n_k \\
&- \left(\frac{\xi_7}{2} - \frac{\xi_9}{2} - \frac{\xi_{11}}{4} \right) n_i n_j n_l n_{k,l} \\
&+ \frac{\xi_{11}}{4} (\delta_{ik} n_{j,l} + \delta_{jk} n_{i,l}) n_l \\
&+ \frac{\xi_{12}}{2} (n_{i,j} + n_{j,i}) n_k.
\end{aligned}$$

The coupling tensor β_{ijk} associated with the viscosity β_{12} is given by

$$\beta_{ijk} = \frac{\beta_{12}}{2} n_l (\epsilon_{ilj} n_k + \epsilon_{ilk} n_j).$$

The coupling tensor v_{ijkl} associated with the viscosities $\beta_{0,2,3}$ is given by

$$\begin{aligned}
v_{ijkl} &= \left(\beta_0 - \beta_2 + \frac{\beta_3}{2} \right) n_i n_j n_k n_l \\
&+ \left(\frac{\beta_2}{4} - \frac{\beta_3}{2} \right) ([n_i \delta_{jk} + n_j \delta_{ik}] n_l + [n_i \delta_{jl} + n_j \delta_{il}] n_k) \\
&+ \frac{\beta_3}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
\end{aligned}$$

B.2 Convention of Pleiner and Brand

The tensor of thermal conductivity κ'_{ij} is given by

$$\kappa'_{ij} = \kappa'_\perp \delta_{ij}^\perp + \kappa'_\parallel n_i n_j.$$

The coupling tensor Π_{ij} associated with all thermomechanical effects is given by

$$\begin{aligned}
\Pi_{ij} &= \frac{v'}{\gamma_1} \delta_{ij}^\perp \\
&+ (\pi_1 n_{p,p} \delta_{ij} + \pi_2 n_{j,l} + \pi_3 n_{l,j} + [\pi_4 - \pi_3] n_p n_j n_{l,p}) \epsilon_{ikl} n_k
\end{aligned}$$

The coupling tensor Ψ_{ijk} associated with all thermohydrodynamic effects is given by

$$\begin{aligned}
\Psi_{ijk} &= -\frac{\mu'}{2} (n_i \epsilon_{jlk} + n_j \epsilon_{ilk}) n_l \\
&+ \psi_1 (n_i \delta_{jk}^\perp + n_j \delta_{ik}^\perp) n_{l,l} \\
&+ (\psi_8 - \psi_2) n_{l,l} n_i n_j n_k \\
&+ \psi_3 (n_i \delta_{ij}^\perp + n_j \delta_{il}^\perp) n_{k,l} \\
&+ (\psi_6 - \psi_4) n_{k,l} n_i n_j \\
&+ \psi_5 (n_i n_{j,l} + n_j n_{i,l}) \delta_{kl}^\perp \\
&+ \psi_7 (n_{j,l} \delta_{il}^\perp + n_{i,l} \delta_{jl}^\perp) n_k \\
&+ \psi_9 (n_{j,l} \delta_{ik}^\perp + n_{i,l} \delta_{jk}^\perp) n_l \\
&+ \psi_{10} (n_i n_{j,l} + n_j n_{i,l}) n_k n_l
\end{aligned}$$

Note that Brand and Pleiner [21] use the notation α_i for the coefficients ψ_i in the previous equation: we changed the notation to avoid a conflict with the Leslie viscosities. Furthermore, in their derivation of thermohydrodynamic effects these authors did not assume incompressibility and found 11 coefficients instead of 9 independent coefficients as in the coupling tensor ζ_{ijk} . However, in the incompressibility regime, there remains only nine independent coefficients as

highlighted in the previous equation, because two terms proportional to $\psi_2 \delta_{ij}$ and $\psi_4 \delta_{ij}$ can be absorbed in the definition of the pressure [1,22,24].

The coupling tensor λ_{ijk} associated with the dimensionless coefficient λ is given by

$$\lambda_{ijk} = \frac{\lambda}{2} n_l (\epsilon_{ijl} n_k + \epsilon_{ikl} n_j)$$

The coupling tensor v'_{ijkl} associated with the viscosities v_{1-3} is given by

$$\begin{aligned}
v'_{ijkl} &= 2(v_1 + v_2 - 2v_3) n_i n_j n_k n_l \\
&+ (v_3 - v_2) ([n_i \delta_{jk} + n_j \delta_{ik}] n_l + [n_i \delta_{jl} + n_j \delta_{il}] n_k) \\
&+ v_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
\end{aligned}$$

Again, out of the five independent coefficients introduced by Pleiner and Brand in the compressible regime [22], there remains only three independent coefficients in the incompressible regime – as in the coupling tensor v_{ijkl} of our formalism.

B.3 Equivalence between the conventions

The equivalence between the viscosities of each convention is given by

$$\begin{array}{l|l}
\beta_1 &= \gamma'_1 \equiv \gamma_1 \\
\beta_{12} &= -\lambda \gamma'_1 \equiv \gamma_2 \\
\beta_0 &= 2v_1 + v_2 \\
\beta_2 &= \gamma'_1 \lambda^2 + 4v_3 \\
\beta_3 &= 2v_2 \\
\gamma'_1 &= \beta_1 \\
\lambda &= -\frac{\beta_{12}}{\beta_1} \\
v_1 &= \frac{\beta_0}{2} - \frac{\beta_3}{4} \\
v_2 &= \frac{\beta_3}{2} \\
v_3 &= \frac{\beta_2}{4} - \frac{(\beta_{12})^2}{\beta_1}
\end{array}$$

The equivalence between the thermomechanical coefficients of each convention is given by

$$\begin{array}{l|l}
v &= v' \\
\xi_1 &= \gamma'_1 (\pi_1 + \frac{\pi_2}{2} + \frac{\pi_3}{2}) \\
\xi_2 &= \frac{\gamma'_1}{2} (\pi_2 - \pi_3) \\
\xi_3 &= \gamma'_1 (\pi_2 + \pi_3) \\
\xi_4 &= 2\gamma'_1 \pi_4 \\
v' &= v \\
\pi_1 &= \frac{1}{\beta_1} (\xi_1 - \frac{\xi_3}{2}) \\
\pi_2 &= \frac{1}{\beta_1} (\xi_2 + \frac{\xi_3}{2}) \\
\pi_3 &= \frac{1}{\beta_1} (-\xi_2 + \frac{\xi_3}{2}) \\
\pi_4 &= \frac{\xi_4}{2\beta_1}
\end{array}$$

Note that this expression was already given in a previous article [28], where the presence of flow was neglected.

The equivalence between the thermohydrodynamic coefficients of each convention is given by

$$\begin{aligned}
\mu &= \mu' - v' \lambda \\
\xi_5 &= -(2\psi_1 + \psi_3 + \psi_5) + \gamma'_1 \lambda (\pi_1 + \frac{\pi_2}{2} + \frac{\pi_3}{2}) \\
\xi_6 &= \psi_5 - \psi_3 - \frac{\gamma'_1 \lambda}{2} (\pi_3 - \pi_2) \\
\xi_7 &= -2(\psi_3 + \psi_5) + \gamma'_1 \lambda (\pi_3 + \pi_2) \\
\xi_8 &= -4\psi_{10} \\
\xi_9 &= 2(\psi_4 - \psi_6 + \psi_9) \\
\xi_{10} &= \psi_2 + \psi_7 - \psi_8 - 2\psi_9 \\
\xi_{11} &= -4\psi_9 \\
\xi_{12} &= -2\psi_7
\end{aligned}$$

or equivalently by

$$\begin{aligned}
\mu' &= \mu - \frac{\beta_{12} v}{\beta_1} \\
\psi_1 &= \frac{\xi_7}{4} - \frac{\xi_5}{2} + \frac{\beta_{12}}{\beta_1} \left(\frac{\xi_3}{4} - \frac{\xi_1}{2} \right) \\
\psi_2 - \psi_8 &= \xi_{10} - \frac{\xi_{11}}{2} + \frac{\xi_{12}}{2} \\
\psi_3 &= -\left(\frac{\xi_6}{2} + \frac{\xi_7}{4} \right) - \frac{\beta_{12}}{\beta_1} \left(\frac{\xi_2}{2} + \frac{\xi_3}{4} \right) \\
\psi_4 - \psi_6 &= \frac{\xi_9}{2} + \frac{\xi_{11}}{4} \\
\psi_5 &= \frac{\xi_6}{2} - \frac{\xi_7}{4} + \frac{\beta_{12}}{\beta_1} \left(\frac{\xi_2}{2} - \frac{\xi_3}{4} \right) \\
\psi_7 &= -\frac{\xi_{12}}{2} \\
\psi_9 &= -\frac{\xi_{11}}{4} \\
\psi_{10} &= -\frac{\xi_8}{4}
\end{aligned}$$

We recall that in the incompressible regime, ψ_2 (ψ_4) cannot be considered independently of ψ_8 (ψ_6), as can be seen in the expression of Ψ_{ijk} given above.

As explained in the main text, the equivalence between the thermal conductivity tensors κ_{ij} and κ'_{ij} can be made exact only by including corrective terms in $\nabla \mathbf{n}$ in these tensors. This is not really a problem since these corrective terms are extremely small, so that we can always assume $\kappa_{\parallel} = \kappa'_{\parallel}$ and $\kappa_{\perp} = \kappa'_{\perp}$ to within an excellent approximation.